Simple Proof of Invariance of the Bargmann-Wigner Scalar Products

Marek Czachor*
Wydział Fizyki Technicznej i Matematyki Stosowanej
Politechnika Gdańska, ul. Narutowicza 11/12, 80-952 Gdańsk, Poland

An explicitly covariant formalism for dealing with Bargmann-Wigner fields is developed. An invariance of the Barmann-Wigner norm can be proved in a unified way for both massive and massless fields. It is shown that there exists some freedom in the choice of the form of the Bargmann-Wigner scalar product.

I. INTRODUCTION

The main objective of this paper is to prove invariance of the Bargamann-Wigner scalar products [1] in a manifestly covariant way. There are several reasons for undertaking this task. One of them is to fill a sort of gap between the powerful covariant spinor methods [2,3] and the noncovariant methods of induced representations of the Poincaré group [4]. The noncovariance of the induced representations, and especially of their generators, manifests itself in a particular decomposition of spinor generators into boosts and rotations which is used to represent boosts by the so-called Wigner rotations [5]. In addition, a transition between Wigner and spinor bases involves dividing a Fourier transform of the spinor field by some powers of energy p^0. Typically, this p^0 is identified with the p^0 appearing in the invariant measure $d^3p/(2|p^0|)$ and leads to the characteristic additional powers $|p^0|^n$ appearing in the spinor versions of the Bargmann-Wigner products. In the covariant form of the scalar product given below these additional powers of energy will be shown to possess some arbitrariness which is normally hidden behind the noncovariance of the standard expressions for the scalar products.

Covariant, and especially spinor methods are known to be a very efficient tool for dealing with relativistic field theories. The methods of Hilbert spaces which, implicitly, are those related to the Bargmann-Wigner scalar products, were shown recently to play an important role in a wavelet formulation of electrodynamics [6]. One may hope that the results presented in this paper will prove useful for the wavelet formulation of higher spin fields.

II. MASSIVE BARGMANN-WIGNER FIELDS

The Bargmann-Wigner equations [1,4,5] representing free spin-$n/2$ fields with mass $m \neq 0$ are equivalent to the set of spinor field equations for 2^n fields $\psi^{0\dots0}_{A_1\dots A_n}$, $\psi^{0\dots1}_{A_1\dots A'_n}$,... $\psi^{1\dots1}_{A'_1\dots A'_n}$,

$$i\nabla^A{}_{A'}\psi(x)^{\dots0\dots}_{\dots A\dots} = -\frac{m}{\sqrt{2}}\psi(x)^{\dots1\dots}_{\dots A'\dots}, \tag{1}$$

$$i\nabla_A{}^{A'}\psi(x)^{\dots1\dots}_{\dots A'\dots} = \frac{m}{\sqrt{2}}\psi(x)^{\dots0\dots}_{\dots A\dots}. \tag{2}$$

The convention we use differs slightly from the one introduced by Penrose and Rindler [2] (see Appendix V B). Let $p^a_\pm = (\pm|p^0|, \boldsymbol{p})$. The Fourier representation of the field is

$$\psi(x)^{\dots}_{\dots} = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2|p^0|}\, e^{ip\cdot x}\Big\{e^{-i|p^0|x^0}\psi_+(\boldsymbol{p})^{\dots}_{\dots} + e^{i|p^0|x^0}\psi_-(\boldsymbol{p})^{\dots}_{\dots}\Big\} \tag{3}$$

where $\psi_\pm(\boldsymbol{p})^{\dots}_{\dots}$ satisfy

$$p_\pm{}^A{}_{A'}\psi_\pm(\boldsymbol{p})^{\dots0\dots}_{\dots A\dots} = -\frac{m}{\sqrt{2}}\psi_\pm(\boldsymbol{p})^{\dots1\dots}_{\dots A'\dots}, \tag{4}$$

$$p_\pm{}_A{}^{A'}\psi_\pm(\boldsymbol{p})^{\dots1\dots}_{\dots A'\dots} = \frac{m}{\sqrt{2}}\psi_\pm(\boldsymbol{p})^{\dots0\dots}_{\dots A\dots}. \tag{5}$$

Consider now the tensor

$$T_\pm(\boldsymbol{p})_{a_1\dots a_n} = \psi_\pm(\boldsymbol{p})^{0\dots0}_{A_1\dots A_n}\bar\psi_\pm(\boldsymbol{p})^{0\dots0}_{A'_1\dots A'_n} + \psi_\pm(\boldsymbol{p})^{0\dots1}_{A_1\dots A'_n}\bar\psi_\pm(\boldsymbol{p})^{0\dots1}_{A'_1\dots A_n} + \dots + \psi_\pm(\boldsymbol{p})^{1\dots1}_{A'_1\dots A'_n}\bar\psi_\pm(\boldsymbol{p})^{1\dots1}_{A_1\dots A_n} \tag{6}$$

$$= \psi_\pm(\boldsymbol{p})^{0\dots0}_{A_1\dots A_n}\overline{\psi_\pm(\boldsymbol{p})^{0\dots0}_{A_1\dots A_n}} + \psi_\pm(\boldsymbol{p})^{0\dots1}_{A_1\dots A'_n}\overline{\psi_\pm(\boldsymbol{p})^{0\dots1}_{A_1\dots A'_n}} + \dots + \psi_\pm(\boldsymbol{p})^{1\dots1}_{A'_1\dots A'_n}\overline{\psi_\pm(\boldsymbol{p})^{1\dots1}_{A'_1\dots A'_n}} \tag{7}$$

where we have, as usual, identified pairs AA' of spinor indices with the world-vector indices a. The standard Bargmann-Wigner scalar product is defined by the norm

$$\parallel \psi_\pm \parallel^2 = \int \frac{d^3p}{2|p^0|^{n+1}} \{\psi_\pm(\boldsymbol{p})^{0\ldots0}_{0\ldots0}\overline{\psi_\pm(\boldsymbol{p})^{0\ldots0}_{0\ldots0}} + \psi_\pm(\boldsymbol{p})^{0\ldots0}_{0\ldots1}\overline{\psi_\pm(\boldsymbol{p})^{0\ldots0}_{0\ldots1}} + \ldots + \psi_\pm(\boldsymbol{p})^{1\ldots1}_{1'\ldots1'}\overline{\psi_\pm(\boldsymbol{p})^{1\ldots1}_{1'\ldots1'}}\} \tag{8}$$

which being invariant under the Poincaré group is not *manifestly* invariant. The lack of the manifest invariance leads to difficulties with applying the spinor methods in the context of induced representations.

To get the manifestly invariant form we shall first rewrite the tensor $T_\pm(\boldsymbol{p})_{a_1\ldots a_n}$ with the help of the field equations as follows

$$T_\pm(\boldsymbol{p})_{a_1\ldots a_k\ldots a_n} = 2m^{-2}p_{\pm A_k}{}^{B'_k}p_\pm{}^{B_k}{}_{A'_k}T_\pm(\boldsymbol{p})_{a_1\ldots b_k\ldots a_n} = 2m^{-2}\Big(p_{\pm a_k}p_\pm^{b_k} - \frac{m^2}{2}g_{a_k}{}^{b_k}\Big)T_\pm(\boldsymbol{p})_{a_1\ldots b_k\ldots a_n}, \tag{9}$$

where we have used the trace-reversal spinor formula [2]

$$p_{AB'}p_{BA'} = p_ap_b - \frac{m^2}{2}g_{ab}. \tag{10}$$

Therefore

$$T_\pm(\boldsymbol{p})_{a_1\ldots a_k\ldots a_n} = m^{-2}p_{\pm a_k}p_\pm^{b_k}T_\pm(\boldsymbol{p})_{a_1\ldots b_k\ldots a_n}. \tag{11}$$

Applying (11) to itself n times we get

$$T_\pm(\boldsymbol{p})_{a_1\ldots a_n} = m^{-2n}p_{\pm a_1}\ldots p_{\pm a_n}p_\pm^{b_1}\ldots p_\pm^{b_n}T_\pm(\boldsymbol{p})_{b_1\ldots b_n}. \tag{12}$$

The Poincaré (i.e. spinor) transformation of the Bargmann-Wigner field implies

$$T'_\pm(\boldsymbol{p})_{a_1\ldots a_n} = m^{-2n}p_{\pm a_1}\ldots p_{\pm a_n}p_\pm^{b_1}\ldots p_\pm^{b_n}T'_\pm(\boldsymbol{p})_{b_1\ldots b_n} \tag{13}$$

$$= m^{-2n}p_{\pm a_1}\ldots p_{\pm a_n}p_\pm^{b_1}\ldots p_\pm^{b_n}\Lambda_{b_1}{}^{c_1}\ldots\Lambda_{b_n}{}^{c_n}T_\pm(\boldsymbol{\Lambda^{-1}p})_{c_1\ldots c_n} \tag{14}$$

$$= m^{-2n}p_{\pm a_1}\ldots p_{\pm a_n}(\Lambda^{-1}p_\pm)^{b_1}\ldots(\Lambda^{-1}p_\pm)^{b_n}T_\pm(\boldsymbol{\Lambda^{-1}p})_{b_1\ldots b_n} \tag{15}$$

Let $t_1^a,\ldots,t_n^a$ be arbitrary world-vectors satisfying $t_k^ap_a \neq 0$ for any p_a belonging to the mass hyperboloid. The expression

$$\parallel \psi_\pm \parallel'^2 = \int \frac{d^3p}{2|p^0|}\frac{t_1^{a_1}\ldots t_n^{a_n}T_\pm(\boldsymbol{p})_{a_1\ldots a_n}}{t_1^{b_1}\ldots t_n^{b_n}p_{\pm b_1}\ldots p_{\pm b_n}} = m^{-2n}\int\frac{d^3p}{2|p^0|}p_\pm^{a_1}\ldots p_\pm^{a_n}T_\pm(\boldsymbol{p})_{a_1\ldots a_n} \tag{16}$$

is manifestly invariant. It is interesting that the LHS of (16) is independent of the choice of $t_1^a,\ldots,t_n^a$ because the RHS does not depend on them. We can take now $t_k^a = t_\pm^a$ where $t_\pm^ap_{\pm a}$ is equal to $|p^0|$ used in the invariant measure. The matrix form of $t_\pm^{AA'}$ is (cf. [2])

$$t_\pm^{\mathbf{AA'}} = t_\pm^a g_a{}^{\mathbf{AA'}} = t_\pm^0 g_0{}^{\mathbf{AA'}} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{17}$$

We have therefore

$$\parallel \psi_\pm \parallel'^2 = (\pm1)^n\int\frac{d^3p}{2|p^0|^{n+1}}T_\pm(\boldsymbol{p})_{0\ldots0} = (\pm1)^n2^{-n/2}\parallel \psi_\pm \parallel^2, \tag{18}$$

and it follows that the Bargmann-Wigner norm can be written in the manifestly invariant form

$$\parallel \psi_\pm \parallel^2 = (\pm1)^n2^{n/2}m^{-2n}\int d\mu_m(\boldsymbol{p})p_\pm^{a_1}\ldots p_\pm^{a_n}T_\pm(\boldsymbol{p})_{a_1\ldots a_n}, \tag{19}$$

where $d\mu_m(\boldsymbol{p})$ is the invariant measure on the mass hyperboloid. In the simplest example of the Dirac equation we find

$$T_\pm(\boldsymbol{p})_a = g_a{}^{AA'}\big(\psi_\pm(\boldsymbol{p})^0_A\bar\psi_\pm(\boldsymbol{p})^0_{A'} + \psi_\pm(\boldsymbol{p})^1_{A'}\bar\psi_\pm(\boldsymbol{p})^1_A\big) = 2^{-1/2}\bar\Psi_\pm(\boldsymbol{p})\gamma_a\Psi_\pm(\boldsymbol{p}), \tag{20}$$

where

$$\Psi_\pm(\boldsymbol{p}) = \begin{pmatrix}\psi_\pm(\boldsymbol{p})^0_A\\ \psi_\pm(\boldsymbol{p})^1_{A'}\end{pmatrix} \tag{21}$$

is the Dirac bispinor, and

$$\parallel \psi_\pm \parallel^2 = \pm m^{-2}\int d\mu_m(\boldsymbol{p})p_\pm^a\bar\Psi_\pm(\boldsymbol{p})\gamma_a\Psi_\pm(\boldsymbol{p}). \tag{22}$$

III. MASSLESS FIELDS

A massless spin-$n/2$ field is described by the spinor equations [2]

$$\nabla^{A_k}{}_{A'_k}\psi(x)_{A_1\dots A_k\dots A_r A'_1\dots A'_{r\pm n}} = 0, \tag{23}$$

$$\nabla_{A_k}{}^{A'_k}\psi(x)_{A_1\dots A_r A'_1\dots A'_k\dots A'_{r\pm n}} = 0, \tag{24}$$

where the spinor $\psi(x)_{A_1\dots A_r A'_1\dots A'_{r\pm n}}$ is totally symmetric in all indices. For simplicity of notation let us consider the case $r = n$, and a field which has only unprimed indices.

With any massless field one can associate various types of potentials [3]. The Hertz-type potentials are defined by

$$\psi(x)_{A_1\dots A_n} = \nabla_{A_1A'_1}\dots\nabla_{A_nA'_n}\xi(x)^{A'_1\dots A'_n} \tag{25}$$

with the subsidiary condition

$$\Box\xi(x)^{A'_1\dots A'_n} = 0. \tag{26}$$

The fact, known generally from the representation theory, that the field $\psi(x)_{A_1\dots A_n}$ carries only one helicity (one degree of freedom) corresponds to the possibility of writing

$$\xi(x)^{A'_1\dots A'_n} = \xi^{A'_1\dots A'_n}\xi(x) \tag{27}$$

where $\xi^{A'_1\dots A'_n}$ is constant and $\Box\xi = 0$.

Potentials of another type are defined by

$$\psi(x)_{A_1\dots A_n} = \nabla_{A_1A'_1}\dots\nabla_{A_kA'_k}\phi(x)^{A'_1\dots A'_k}_{A_{k+1}\dots A_n}, \tag{28}$$

and are subject to

$$\nabla^{A_{k+1}A'_{k+1}}\phi(x)^{A'_1\dots A'_k}_{A_{k+1}\dots A_n} = 0 \tag{29}$$

implying the generalized Lorenz gauge

$$\nabla^{A_{k+1}}{}_{A'_k}\phi(x)^{A'_1\dots A'_k}_{A_{k+1}\dots A_n} = 0. \tag{30}$$

Let us begin with the Fourier representation of both the spinor field and its Hertz-type potential:

$$\psi(x)_{A_1\dots A_n} = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2|p^0|}\,e^{ip\cdot x}\Big\{e^{-i|p^0|x^0}\psi_+(\boldsymbol{p})_{A_1\dots A_n} + e^{i|p^0|x^0}\psi_-(\boldsymbol{p})_{A_1\dots A_n}\Big\} \tag{31}$$

$$\xi(x)^{A'_1\dots A'_n} = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2|p^0|}\,e^{ip\cdot x}\Big\{e^{-i|p^0|x^0}\xi_+(\boldsymbol{p})^{A'_1\dots A'_n} + e^{i|p^0|x^0}\xi_-(\boldsymbol{p})^{A'_1\dots A'_n}\Big\}. \tag{32}$$

These definitions imply that

$$\psi_\pm(\boldsymbol{p})_{A_1\dots A_n} = (-i)^n p_{\pm A_1A'_1}\dots p_{\pm A_nA'_n}\xi_\pm(\boldsymbol{p})^{A'_1\dots A'_n}. \tag{33}$$

The rest of the construction is analogous to the massive case. We define the tensor

$$\begin{aligned} T_\pm(\boldsymbol{p})_{a_1\dots a_n} &= \psi_\pm(\boldsymbol{p})_{A_1\dots A_n}\bar\psi_\pm(\boldsymbol{p})_{A'_1\dots A'_n} && (34)\\ &= p_{\pm A_1B'_1}p_{\pm B_1A'_1}\dots p_{\pm A_nB'_n}p_{\pm B_nA'_n}\xi_\pm(\boldsymbol{p})^{B'_1\dots B'_n}\bar\xi_\pm(\boldsymbol{p})^{B_1\dots B_n} \\ &= p_{\pm a_1}\dots p_{\pm a_n}p_{\pm b_1}\dots p_{\pm b_n}U_\pm(\boldsymbol{p})^{b_1\dots b_n}, && (35)\end{aligned}$$

where

$$U_\pm(\boldsymbol{p})^{b_1\dots b_n} = \xi_\pm(\boldsymbol{p})^{B'_1\dots B'_n}\bar\xi_\pm(\boldsymbol{p})^{B_1\dots B_n}. \tag{36}$$

Similarly to the massive case we define

$$\parallel \psi_\pm \parallel'^2 = \int \frac{d^3p}{2|p^0|} \frac{t_1^{a_1}\dots t_n^{a_n} T_\pm(\boldsymbol{p})_{a_1\dots a_n}}{t_1^{b_1}\dots t_n^{b_n} p_{\pm b_1}\dots p_{\pm b_n}} = \int \frac{d^3p}{2|p^0|} p_\pm^{a_1}\dots p_\pm^{a_n} U_\pm(\boldsymbol{p})_{a_1\dots a_n} \tag{37}$$

which is manifestly invariant and independent of the choice of $t_1^a, \dots, t_n^a$. The expression (37) is directly related to the Bargmann-Wigner norm. But to see this we first have to make the one-dimensionality of the representation explicit.

The well known fact that the field

$$\psi(x)_{A_1\dots A_r A'_1 \dots A'_{r\pm n}} \tag{38}$$

carries only one helicity can be shown in a covariant manner as follows. We first contract the field equation (23) with $g^a{}_{A_k A'_k}$ and use the identity (cf. Appendix V A)

$$g^a{}_{XA'} g^{bYA'} = \frac{1}{2} g^{ab} \varepsilon_X{}^Y + i\sigma^{ab}{}_X{}^Y \tag{39}$$

where $\sigma^{ab}{}_X{}^Y$ is the generator of the $(1/2,0)$ spinor representation. Performing an analogous transformation of (24), denoting $P^a = i\nabla^a$, and introducing the Pauli-Lubanski tensors corresponding to $(1/2,0)$ and $(0,1/2)$ representations by

$$S^a{}_X{}^Y = P_b{}^* \sigma^{ba}{}_X{}^Y, \tag{40}$$

$$S^a{}_{X'}{}^{Y'} = P_b{}^* \bar\sigma^{ba}{}_{X'}{}^{Y'}, \tag{41}$$

we obtain the equivalent form of (23) and (24)

$$-\frac{1}{2} P^a \psi(x)_{A_1\dots A_r A'_1\dots A'_{r\pm n}} = S^a{}_{A_k}{}^{B_k} \psi(x)_{A_1\dots B_k \dots A_r A'_1 \dots A'_{r\pm n}}, \tag{42}$$

$$\frac{1}{2} P^a \psi(x)_{A_1\dots A_r A'_1\dots A'_{r\pm n}} = S^a{}_{A'_k}{}^{B'_k} \psi(x)_{A_1\dots A_r A'_1 \dots B'_k \dots A'_{r\pm n}}. \tag{43}$$

We can further simplify the equations by introducing the generators $\sigma^{ab}{}_{\mathcal{A}}{}^{\mathcal{B}}$ of the $(r/2, r/2 \pm n/2)$ representation. With the help of the respective Pauli-Lubanski vector the massless equation reduces to

$$\pm \frac{n}{2} P^a \psi(x)_{\mathcal{A}} = S^a{}_{\mathcal{A}}{}^{\mathcal{B}} \psi(x)_{\mathcal{B}}, \tag{44}$$

where $\mathcal{A}$, $\mathcal{B}$ stand for $A_1 \dots A_r A'_1 \dots A'_{r\pm n}$, etc.

At the level of the Fourier transform the one-dimensionality of the representation follows immediately from the Hertz-type form of the potentials. Indeed, the momentum representation of the Pauli-Lubanski vector is

$$-\frac{1}{2}\Big(p_{\pm XA'} g^{aYA'} - g^a{}_{XA'} p_\pm^{YA'} \Big) = S^a_\pm(\boldsymbol{p})_X{}^Y, \tag{45}$$

$$\frac{1}{2}\Big(p_{\pm AX'} g^{aAY'} - g^a{}_{AX'} p_\pm^{AY'} \Big) = S^a_\pm(\boldsymbol{p})_{X'}{}^{Y'}. \tag{46}$$

Using the trace-reversal formula, the identity

$$p_{AA'} p^{AB'} = \frac{1}{2} p_a p^a \, \varepsilon_{A'}{}^{B'}, \tag{47}$$

and its complex-conjugated version, we get

$$S^a_\pm(\boldsymbol{p})_X{}^Y p_{\pm YX'} = -\frac{1}{2} p^a_\pm \, p_{\pm XX'}, \tag{48}$$

$$S^a_\pm(\boldsymbol{p})_{X'}{}^{Y'} p_{\pm XY'} = \frac{1}{2} p^a_\pm \, p_{\pm XX'}, \tag{49}$$

which imply (44) which means that the spinor

$$\xi_\pm(\boldsymbol{p})^{A'_1\dots A'_n} \tag{50}$$

in (33) is in fact arbitrary. The eigenequation (44) determines the Fourier components of the field up to a $\boldsymbol{p}$-dependent factor (an "amplitude"). We can write, therefore,

$$\psi_\pm(\boldsymbol{p})_{A_1\dots A_n} = (-i)^n p_{\pm A_1 A'_1}\dots p_{\pm A_n A'_n}\eta_\pm(\boldsymbol{p})^{A'_1\dots A'_n} f_\pm(\boldsymbol{p}), \tag{51}$$

where the only restriction on $f_\pm(\boldsymbol{p})$ is the square-integrability of the field, and $\eta_\pm(\boldsymbol{p})^{A'_1\dots A'_n}$ is normalized by

$$p_{\pm b_1}\dots p_{\pm b_n}\eta_\pm(\boldsymbol{p})^{B'_1\dots B'_n}\bar{\eta}_\pm(\boldsymbol{p})^{B_1\dots B_n} = (\pm 1)^n. \tag{52}$$

We can choose $\eta_\pm(\boldsymbol{p})^{A'_1\dots A'_n}$ as follows. Let $p_{\pm a} = \pm\pi_{\pm A}\,\bar{\pi}_{\pm A'}$, and let $\omega^A_\pm$ satisfy $\pi_{\pm A}\omega^A_\pm = 1$ (i.e. the pair $\pi_{\pm A}$, $\omega^A_\pm$ is a spin-frame [2]). Then

$$\eta_\pm(\boldsymbol{p})^{A'_1\dots A'_n} = \bar{\omega}^{A'_1}_\pm\dots\bar{\omega}^{A'_n}_\pm, \tag{53}$$

and

$$\psi_\pm(\boldsymbol{p})_{A_1\dots A_n} = (\mp i)^n\pi_{\pm A_1}\dots\pi_{\pm A_n} f_\pm(\boldsymbol{p}). \tag{54}$$

The amplitude then satisfies

$$(\pm 1)^n \parallel \psi_\pm \parallel'^2 = \int \frac{d^3p}{2|p^0|}\,|f_\pm(\boldsymbol{p})|^2. \tag{55}$$

Therefore $f_\pm(\boldsymbol{p})$ is the Bargmann-Wigner amplitude which is used in [7,8,9,10] in the context of the electromagnetic field and the photon wave function. The form (54) resembles kernels of contour integral expressions for massless fields arising in the twistor formalism (cf. [3], Eq. (6.10.3) on p. 140), and shows that the Bargmann-Wigner amplitude is closely related to twistor wave functions.

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V. APPENDICES

A. Infeld-van der Waerden tensors and generators of (1/2,0) and (0,1/2)

Consider representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of an element $\omega\in SL(2,C)$: $e^{\frac{i}{2}\omega^{ab}\sigma_{ab}}$ and $e^{\frac{i}{2}\omega^{ab}\bar{\sigma}_{ab}}$. The explicit form of the generators in terms of Infeld-van der Waerden tensors is

$$\frac{1}{2i}\Big(g^a{}_{XA'}g^{bYA'} - g^b{}_{XA'}g^{aYA'}\Big) = \sigma^{ab}{}_X{}^Y, \tag{56}$$

$$\frac{1}{2i}\Big(g^a{}_{AX'}g^{bAY'} - g^b{}_{AX'}g^{aAY'}\Big) = \bar{\sigma}^{ab}{}_{X'}{}^{Y'}. \tag{57}$$

Their purely spinor form is

$$\sigma_{AA'BB'XY} = \frac{1}{2i}\varepsilon_{A'B'}\big(\varepsilon_{AX}\varepsilon_{BY} + \varepsilon_{BX}\varepsilon_{AY}\big), \tag{58}$$

$$\bar{\sigma}_{AA'BB'X'Y'} = \frac{1}{2i}\varepsilon_{AB}\big(\varepsilon_{A'X'}\varepsilon_{B'Y'} + \varepsilon_{B'X'}\varepsilon_{A'Y'}\big), \tag{59}$$

Dual tensors are ${}^*\bar{\sigma}^{ab}{}_{X'}{}^{Y'} = +i\bar{\sigma}^{ab}{}_{X'}{}^{Y'}$ and ${}^*\sigma^{ab}{}_X{}^Y = -i\sigma^{ab}{}_X{}^Y$.

Additionally the Infeld-van der Waerden tensors satisfy

$$g^a{}_{XA'}g^{bYA'} + g^b{}_{XA'}g^{aYA'} = g^{ab}\varepsilon_X{}^Y \tag{60}$$

$$g^a{}_{AX'}g^{bAY'} + g^b{}_{AX'}g^{aAY'} = g^{ab}\varepsilon_{X'}{}^{Y'} \tag{61}$$

These equations lead to the useful expressions

$$g^a{}_{XA'}g^{bYA'} = \frac{1}{2}g^{ab}\varepsilon_X{}^Y + i\sigma^{ab}{}_X{}^Y \tag{62}$$

$$g^a{}_{AX'}g^{bAY'} = \frac{1}{2}g^{ab}\varepsilon_{X'}{}^{Y'} + i\bar{\sigma}^{ab}{}_{X'}{}^{Y'} \tag{63}$$

B. Spinor and bispinor forms of the Dirac equation

The matrix form of the Dirac in the momentum representation equation can be written explicitly as

$$\begin{pmatrix} 0 & (p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma})^{AB'} \\ (p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma})_{A'B} & 0 \end{pmatrix} \begin{pmatrix} \psi^B \\ \xi_{B'} \end{pmatrix} = \begin{pmatrix} 0 & p^a {\sigma_a}^{AB'} \\ p^a \tilde{\sigma}_{aA'B} & 0 \end{pmatrix} \begin{pmatrix} \psi^B \\ \xi_{B'} \end{pmatrix} = m \begin{pmatrix} \psi^A \\ \xi_{A'} \end{pmatrix}, \tag{64}$$

where

$$ {\sigma_a}^{AB'} = (\mathbf{1}, \boldsymbol{\sigma})^{AB'} \tag{65}$$

$$ \tilde{\sigma}_{aA'B} = (\mathbf{1}, -\boldsymbol{\sigma})_{A'B}, \tag{66}$$

and $\boldsymbol{\sigma}$ is a matrix vector whose components are the Pauli matrices. The matrix formulas

$$\tilde{\sigma}_a \sigma_b + \tilde{\sigma}_b \sigma_a = 2 g_{ab} \mathbf{1} \tag{67}$$

$$\sigma_a \tilde{\sigma}_b + \sigma_b \tilde{\sigma}_a = 2 g_{ab} \mathbf{1} \tag{68}$$

have the following spinor form

$$\tilde{\sigma}_{aA'X} {\sigma_b}^{XB'} + \tilde{\sigma}_{bA'X} {\sigma_a}^{XB'} = 2 g_{ab} {\varepsilon_{A'}}^{B'} \tag{69}$$

$${\sigma_a}^{BX'} \tilde{\sigma}_{bX'A} + {\sigma_b}^{BX'} \tilde{\sigma}_{aX'A} = 2 g_{ab} {\varepsilon_A}^{B} \tag{70}$$

which compared with (60), (61) shows that

$$g_{aAB'} = \frac{1}{\sqrt{2}} \tilde{\sigma}_{aB'A} \tag{71}$$

$${g_a}^{AB'} = \frac{1}{\sqrt{2}} {\sigma_a}^{AB'} \tag{72}$$

The Dirac equation in the Minkowski representation is ($\hbar = 1$)

$$i \nabla_{AA'} \psi^A = \frac{m}{\sqrt{2}} \xi_{A'}, \tag{73}$$

$$i \nabla^{AA'} \xi_{A'} = \frac{m}{\sqrt{2}} \psi^A \tag{74}$$

where $\nabla_{AA'} = \nabla^a g_{aAA'}$ etc. This equation differs by a sign and the presence of i from the form given in [2]. The matrix form of the equation

$$p^q \begin{pmatrix} 0 & {g_{qA}}^{B'} \\ -{g_q}^{B}{}_{A'} & 0 \end{pmatrix} \begin{pmatrix} \psi_B \\ \xi_{B'} \end{pmatrix} = \frac{m}{\sqrt{2}} \begin{pmatrix} \psi_A \\ \xi_{A'} \end{pmatrix} \tag{75}$$

shows that the Dirac gamma matrices are given by

$${\gamma_{q\alpha}}^{\beta} = \sqrt{2} \begin{pmatrix} 0 & {g_{qA}}^{B'} \\ -{g_q}^{B}{}_{A'} & 0 \end{pmatrix} \tag{76}$$

Product of two gamma matrices

$${\gamma_{q\alpha}}^{\beta} {\gamma_{r\beta}}^{\gamma} = \begin{pmatrix} g_{qr} {\varepsilon_A}^C + 2i {\sigma_{qrA}}^C & 0 \\ 0 & g_{qr} {\varepsilon_{A'}}^{C'} + 2i {\bar{\sigma}_{qrA'}}{}^{C'} \end{pmatrix} = g_{qr} {I_\alpha}^\gamma + 2i {\sigma_{qr\alpha}}^\gamma \tag{77}$$

implies

$${\gamma_{q\alpha}}^{\beta} {\gamma_{r\beta}}^{\gamma} + {\gamma_{r\alpha}}^{\beta} {\gamma_{q\beta}}^{\gamma} = 2 g_{qr} {I_\alpha}^\gamma, \tag{78}$$

$${\gamma_{q\alpha}}^{\beta} {\gamma_{r\beta}}^{\gamma} - {\gamma_{r\alpha}}^{\beta} {\gamma_{q\beta}}^{\gamma} = 4i {\sigma_{qr\alpha}}^\gamma. \tag{79}$$

(79) differs by the factor $(-1/2)$ from the definition from [11] because there the generators are defined by $S(\omega) = e^{-\frac{i}{4}\omega^{ab}\sigma_{ab}}$. There is also a difference with respect to [2] where the gamma matrices are defined without the $-$ sign (this would lead to the opposite sign at the RHS of (78)).

The spinor form of the Dirac current is

$$j_a = \sqrt{2} g_a{}^{AA'} \big(\psi_A \bar{\psi}_{A'} + \xi_{A'} \bar{\xi}_A \big). \tag{80}$$

(80) is derived spinorially as follows

$$j_a = \sqrt{2} (\bar{\psi}^{A'}, \bar{\xi}^{A}) \begin{pmatrix} 0 & \varepsilon_{A'}{}^{B'} \\ -\varepsilon_A{}^B & 0 \end{pmatrix} \begin{pmatrix} 0 & g_{aB}{}^{C'} \\ -g_a{}^C{}_{B'} & 0 \end{pmatrix} \begin{pmatrix} \psi_C \\ \xi_{C'} \end{pmatrix} \tag{81}$$

and

$$\begin{aligned} j_0 &= (\bar{\psi}^{A'}, \bar{\xi}^{A}) \underbrace{\begin{pmatrix} 0 & \varepsilon_{A'}{}^{B'} \\ -\varepsilon_A{}^B & 0 \end{pmatrix}}_{\text{“}\gamma_0\text{”}} \underbrace{\sqrt{2} \begin{pmatrix} 0 & g_{0B}{}^{C'} \\ -g_0{}^C{}_{B'} & 0 \end{pmatrix}}_{\text{“}\gamma_0\text{”}} \begin{pmatrix} \psi_C \\ \xi_{C'} \end{pmatrix} \\ &= \sqrt{2} (\bar{\psi}_{A'}, \bar{\xi}_{A}) \begin{pmatrix} g_0{}^{AA'} & 0 \\ 0 & g_0{}^{AA'} \end{pmatrix} \begin{pmatrix} \psi_A \\ \xi_{A'} \end{pmatrix} \end{aligned} \tag{82}$$

showing that the matrix γ_0 appearing in textbooks corresponds actually to two different spinor objects. The pseudoscalar matrix γ_5 corresponds to the spinor matrix

$$\gamma_{5X}{}^{Y} = \frac{i}{4!} e^{abcd} \gamma_a \gamma_b \gamma_c \gamma_{dX}{}^{Y} = \begin{pmatrix} -\varepsilon_X{}^Y & 0 \\ 0 & \varepsilon_{X'}{}^{Y'} \end{pmatrix}. \tag{83}$$

C. Alternative covariant proof for the Maxwell field

The other form of potentials is not very helpful in proving invariance of the Bargmann-Wigner norm in the general spin case. It is instructive, however, to see how the spinor language simplifies the standard proof in the particular case of the Maxwell field (cf. [12] and [6]).

Consider the electromagnetic spinor

$$\varphi_\pm(\boldsymbol{p})_{AB} = \frac{i}{2} F_\pm^{qr}(\boldsymbol{p}) \sigma_{qrAB}, \tag{84}$$

which satisfies

$$\varphi_\pm(\boldsymbol{p})_{AB} = -i p_{\pm AA'} \phi_\pm(\boldsymbol{p})_B{}^{A'} = -i p_{\pm BA'} \phi_\pm(\boldsymbol{p})_A{}^{A'} \tag{85}$$

implying the Lorenz gauge

$$p_{\pm AA'} \phi_\pm(\boldsymbol{p})^{AA'} = 0 \tag{86}$$

for the 4-vector potential $\phi_\pm^a(\boldsymbol{p})$. We consider the tensor

$$\begin{aligned} T_{\pm ab}(\boldsymbol{p}) &= \varphi_\pm(\boldsymbol{p})_{AB} \bar{\varphi}_\pm(\boldsymbol{p})_{A'B'} = p_{\pm AC'} \phi_\pm(\boldsymbol{p})_B{}^{C'} p_{\pm CA'} \phi_\pm(\boldsymbol{p})^C{}_{B'} = p_{\pm AA'} p_{\pm CC'} \phi_\pm(\boldsymbol{p})_B{}^{C'} \phi_\pm(\boldsymbol{p})^C{}_{B'} \\ &= p_{\pm AA'} p_{\pm BC'} \phi_\pm(\boldsymbol{p})_C{}^{C'} \phi_\pm(\boldsymbol{p})^C{}_{B'} = -\frac{1}{2} p_{\pm a} p_{\pm b} \phi_{\pm c}(\boldsymbol{p}) \phi_\pm^c(\boldsymbol{p}), \end{aligned} \tag{87}$$

where we have used the trace-reversal identity and the fact that

$$\phi_\pm(\boldsymbol{p})_{CC'} \phi_\pm(\boldsymbol{p})^C{}_{B'} = -\phi_\pm(\boldsymbol{p})_{CB'} \phi_\pm(\boldsymbol{p})^C{}_{C'}. \tag{88}$$

The tensor satisfies the formula

$$T_{\pm ab}(\boldsymbol{p}) = \frac{1}{2} \Big(\frac{1}{4} g_{ab} F_{\pm cd}(\boldsymbol{p}) F_\pm^{cd}(\boldsymbol{p}) - F_{\pm ac}(\boldsymbol{p}) F_{\pm b}{}^c(\boldsymbol{p}) \Big), \tag{89}$$

and, in particular,

$$T_{\pm 00}(\boldsymbol{p}) = \frac{1}{4}\Big(\boldsymbol{E}_\pm(\boldsymbol{p})^2 + \boldsymbol{B}_\pm(\boldsymbol{p})^2\Big), \tag{90}$$

where $\boldsymbol{E}_\pm(\boldsymbol{p})$ and $\boldsymbol{B}_\pm(\boldsymbol{p})$ are the positive and negative frequency Fourier transforms of the electromagnetic field.

Now we can repeat the reasoning presented above for the general case and the norm used in the wavelet analysis of the electromagnetic field [6] becomes a particular case of

$$\parallel \varphi \parallel^2 = \parallel \varphi_+ \parallel'^2 + \parallel \varphi_- \parallel'^2, \tag{91}$$

where

$$\parallel \varphi_\pm \parallel'^2 = \int \frac{d^3p}{2|p^0|} \frac{t_1^a t_2^b T_{\pm ab}(\boldsymbol{p})}{t_1^c t_2^d p_c p_d}. \tag{92}$$

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* Electronic address: mczachor@sunrise.pg.gda.pl

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